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1976 J. Phys. A: Math. Gen. 9 255

(http://iopscience.iop.org/0305-4470/9/2/010)

J. Phys. A: Math. Gen., Vol. 9, No. 2, 1976. Printed in Great Britain. © 1976

A cylindrically symmetric expanding universe

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Received 21 February 1975, in final form 19 May 1975

Abstract. Considering the cylindrically symmetric metric of Marder, a cosmological model has been derived which is of Petrov type I. Various physical and geometrical properties of the model have been discussed.

1. Introduction

In recent years there has been a lot of interest in cosmological models which are non-isotropic and non-homogeneous. A plane symmetric cosmological model has been constructed by Singh and Singh (1968). Further work in this line has been done by Singh and Abdussattar (1973). In this paper we construct a cosmological model which is cylindrically symmetric and of non-degenerate Petrov type I. The energy–momentum tensor has been assumed to be that of a perfect fluid. Reality conditions imply that the cosmological constant should always be negative. The model is not a particular case of a Lemaître universe. It represents an expanding and shearing but non-rotating fluid flow which is also geodesic. The model becomes conformal to flat space–time in particular cases. The expression for the generalized Doppler effect in the model has been obtained.

2. Derivation of the line element

The cylindrically symmetric metric is considered in the form given by Marder (1958):

$$ds^2 = A^2(dt^2 - dx^2) - B^2\, dy^2 - C^2\, dz^2 \tag{2.1}$$

where A, B, C are functions of t only. The energy–momentum tensor for perfect fluid distribution is given by

$$T_{ij} = (\rho + p)\lambda_i\lambda_j - pg_{ij} \tag{2.2}$$

where

$$g_{ij}\lambda^i\lambda^j = 1. \tag{2.3}$$

The coordinates are assumed to be co-moving so that $\lambda^1 = \lambda^2 = \lambda^3 = 0$.
The field equations

$$R_{ij} - \tfrac{1}{2}Rg_{ij} + \Lambda g_{ij} = -8\pi T_{ij} \qquad \text{(with } C = G = 1\text{)}$$

for the line element (2.1) are as follows:

$$\frac{1}{A^2}\left[\left(\frac{B_{44}}{B}+\frac{C_{44}}{C}\right)-\frac{A_4}{A}\left(\frac{B_4}{B}+\frac{C_4}{C}\right)+\frac{B_4C_4}{BC}\right]+\Lambda = -8\pi p \tag{2.4}$$

$$\frac{1}{A^2}\left[\left(\frac{A_4}{A}\right)_4+\frac{C_{44}}{C}\right]+\Lambda = -8\pi p \tag{2.5}$$

$$\frac{1}{A^2}\left[\left(\frac{A_4}{A}\right)_4+\frac{B_{44}}{B}\right]+\Lambda = -8\pi p \tag{2.6}$$

$$\frac{1}{A^2}\left[\frac{A_4}{A}\left(\frac{B_4}{B}+\frac{C_4}{C}\right)+\frac{B_4C_4}{BC}\right]-\Lambda = 8\pi\rho. \tag{2.7}$$

The non-vanishing components of the Weyl conformal curvature tensor C_{hijk} for the metric (2.1) are as follows:

$$\begin{aligned}
C^{14}_{14} = C^{23}_{23} &= \frac{1}{6A^2}\left[\frac{B_{44}}{B}+\frac{C_{44}}{C}-2\left(\frac{A_4}{A}\right)_4-2\frac{B_4C_4}{BC}\right]\\
C^{12}_{12} = C^{34}_{34} &= \frac{1}{6A^2}\left[\frac{B_{44}}{B}-2\frac{C_{44}}{C}+\left(\frac{A_4}{A}\right)_4+3\frac{A_4}{A}\left(\frac{C_4}{C}-\frac{B_4}{B}\right)+\frac{B_4C_4}{BC}\right]\\
C^{13}_{13} = C^{24}_{24} &= \frac{1}{6A^2}\left[\left(\frac{A_4}{A}\right)_4+\frac{C_{44}}{C}-2\frac{B_{44}}{B}+3\frac{A_4}{A}\left(\frac{B_4}{B}-\frac{C_4}{C}\right)+\frac{B_4C_4}{BC}\right].
\end{aligned} \tag{2.8}$$

Equations (2.4)–(2.7) are four equations in five unknowns A, B, C, ρ and p. For the complete determination of these unknowns one more condition has to be imposed on them. Here we assume that $C^{14}_{14} = C^{23}_{23} = 0$. The resulting space–time will obviously be of non-degenerate Petrov type I. Thus we have

$$\left(\frac{A_4}{A}\right)_4 = \frac{B_{44}}{2B}+\frac{C_{44}}{2C}-\frac{B_4C_4}{BC}. \tag{2.9}$$

From (2.4), (2.5) and (2.6) we obtain

$$\frac{B_{44}}{B} = \frac{C_{44}}{C} \tag{2.10}$$

and

$$\left(\frac{A_4}{A}\right)_4+\frac{A_4}{A}\left(\frac{B_4}{B}+\frac{C_4}{C}\right) = \frac{B_{44}}{B}+\frac{B_4C_4}{BC}. \tag{2.11}$$

Equations (2.9) and (2.11) give

$$\frac{A_4}{A}\left(\frac{B_4}{B}+\frac{C_4}{C}\right) = 2\frac{B_4C_4}{BC}. \tag{2.12}$$

From (2.9) and (2.10) we have

$$\left(\frac{A_4}{A}\right)_4 = \frac{B_{44}}{B}-\frac{B_4C_4}{BC}. \tag{2.13}$$

Equation (2.10) gives

$$\left(\frac{B}{C}\right)_4 = \frac{K}{C^2} \tag{2.14}$$

where K is an arbitrary constant of integration.

Let $B/C = \mu$, $BC = \nu$ so that $B^2 = \mu\nu$ and $C^2 = \nu/\mu$. We have from (2.12)

$$\frac{A_4}{A} = \frac{1}{2}\frac{\nu_4}{\nu} - \frac{1}{2}\frac{K^2}{\nu\nu_4}. \tag{2.15}$$

From (2.13) we have

$$\left(\frac{A_4}{A}\right)_4 = \frac{1}{2}\frac{\nu_{44}}{\nu} - \frac{1}{2}\left(\frac{\nu_4^2}{\nu^2} - \frac{K^2}{\nu^2}\right). \tag{2.16}$$

Equations (2.15) and (2.16) give

$$\left(\frac{1}{\nu\nu_4}\right)_4 + \frac{1}{\nu^2} = 0,$$

so that

$$\nu = \alpha t + \beta \tag{2.17}$$

where α and β are arbitrary constants of integration. From (2.15) we have

$$A = L(\alpha t + \beta)^{(1-a^2)/2} \tag{2.18}$$

where $a = K/\alpha$ and L is an arbitrary constant of integration. Since

$$\frac{\mu_4}{\mu} = \frac{K}{\alpha t + \beta},$$

then

$$\mu = M(\alpha t + \beta)^a \tag{2.19}$$

where M is an arbitrary constant of integration.

We have from (2.17) and (2.19)

$$B^2 = M(\alpha t + \beta)^{1+a} \tag{2.20}$$

and

$$C^2 = \frac{1}{M}(\alpha t + \beta)^{1-a}. \tag{2.21}$$

Therefore, the metric (2.1) admitting perfect fluid distribution reduces to

$$ds^2 = L^2(\alpha t + \beta)^{1-a^2}(dt^2 - dx^2) - M(\alpha t + \beta)^{1+a}\, dy^2 - \frac{1}{M}(\alpha t + \beta)^{1-a}\, dz^2. \tag{2.22}$$

The transformation

$$\alpha t+\beta \to t, \qquad \alpha x \to x, \qquad M^{1/2}y \to y, \qquad M^{-1/2}z \to z, \qquad \alpha^{-1}L \to L$$

reduces (2.22) to the much simpler form

$$ds^2 = L^2 t^{1-a^2}(dt^2 - dx^2) - t^{1+a}\,dy^2 - t^{1-a}\,dz^2. \tag{2.23}$$

3. Some physical and geometrical features

The pressure and density in the model (2.23) are given by

$$8\pi p = \frac{3}{4L^2}(1-a^2)t^{a^2-3} + \Lambda \tag{3.1}$$

and

$$8\pi\rho = \frac{3}{4L^2}(1-a^2)t^{a^2-3} - \Lambda. \tag{3.2}$$

The reality conditions $\rho > 0$, $p > 0$, $\rho \geqslant 3p$ lead to

$$\Lambda < 0 \tag{3.3}$$

and

$$-\Lambda \leqslant \frac{3}{4}\frac{(1-a^2)}{L^2}t^{a^2-3} \leqslant -2\Lambda. \tag{3.4}$$

If $a = 0$ the metric (2.23) becomes conformal to flat space–time. Also if $a = \pm 1$, space–time becomes flat. From (3.4) it is clear that the model exists during a finite interval of time $t_1 \leqslant t \leqslant t_2$.

The flow vector λ^i of the distribution is given by

$$\lambda^1 = \lambda^2 = \lambda^3 = \lambda_1 = \lambda_2 = \lambda_3 = 0$$
$$\lambda^4 = \frac{1}{L}t^{(a^2-1)/2}, \qquad \lambda_4 = Lt^{(1-a^2)/2}. \tag{3.5}$$

Clearly $\lambda^i_{;j}\lambda^j = 0$, so that the flow is geodesic.

The redshift in the model is given by

$$\frac{\lambda+\delta\lambda}{\lambda} = \frac{Lt^{(1-a^2)/2}(t_1^{1+a}+U_z)}{t_2^{1+a}(t^{1-a^2}-U^2)} \tag{3.6}$$

where U is the velocity of the source at the time of emission and U_z is the z component of the velocity.

The scalar of expansion Θ is given by

$$\Theta = \frac{1}{6L}(3-a^2)t^{(a^2-3)/2}. \tag{3.7}$$

The tensor of rotation W_{ij} given by $W_{ij} = \lambda_{i;j} - \lambda_{j;i}$ is zero. Thus the fluid filling the universe is non-rotational.

The components of the shear tensor σ_{ij} defined by

$$\sigma_{ij} = \lambda_{i;j} - \Theta(g_{ij} - \lambda_i \lambda_j)$$

are

$$\sigma_{11} = \frac{3-a^2}{6L} t^{-(1+a^2)/2}$$

$$\sigma_{22} = \frac{3-a^2}{6L} t^{(a^2+2a-1)/2}$$

$$\sigma_{33} = \frac{3-a^2}{6L} t^{(a^2-2a-1)/2} \tag{3.8}$$

$$\sigma_{44} = 0.$$

For $a^2 = 3$, there is no expansion, rotation or shear. However, this case does not correspond to a realistic distribution.

Since $\Lambda < 0$ and $W_{ij} = 0$, Raychaudhuri's (1955, 1957) equation shows that the universe had a singularity in the finite past. However, since the model itself exists in a finite extent of time, this singularity does not occur. We may view the universe represented by this model to be evolved from an earlier stage at $t = t_1$, in which radiation was in collisional equilibrium with matter and they together are represented by a perfect fluid for which $p = \frac{1}{3}\rho$. At $t = t_2$ it goes over to another model in which matter becomes tenuous so that p is zero, the universe itself monotonically expanding all the time.

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